M435
Course Guide

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1.1

Introduction

This course consists of two halves: Metric and Topological Spaces (abbreviated as MTS) and Geometric Topology (abbreviated as GT). The teaching material in one half is completely independent of the material in the other, so that the two halves form separate mini-courses which between them provide a good introduction to the subject of topology. The course was originally part of the third-level course *M335* and was subsequently renumbered as *M386*. (Students were able to select for study any two of four such mini-courses, so that each of these had to be written to be entirely independent.) It is now being presented at fourth level, without any change to the material, as part of a suite of courses, offered on an occasional basis and designed to extend your choice of advanced mathematical topics beyond those offered at third level.

In many universities, the material covered by this course is similarly split into two courses without significant connections being made between the two. But there are major connections, and the subjects could be taught in a much more integrated way. You might be interested in seeing how this can be done by looking at the book by Armstrong in the Further Reading Section at the end of GT *Unit 4*. This book has been republished by Springer (1983).

One of the concepts common to the mini-courses is that of continuity. As the mini-courses were written to be entirely independent, authors took different attitudes towards the definitions to be used for continuity. The definitions are, however, perfectly compatible. The definition used by the vast majority of mathematicians is the one used in MTS, and all assessment of continuity, in both the TMAs and the examination, is confined to the MTS half of the course; the GT definitions are never assessed.

Course Material

Most of the material for this course carries the course code *M335*, rather than *M435*. This is because both halves originally formed part of the course *M335*. We hope that this will not cause any confusion.

Metric and Topological Spaces

8 Units, as follows.

Unit	Title
1	Sets and Functions
2	Some Real Analysis
3	Metric Spaces and Continuity
4	Topological Spaces I
5	Topological Spaces II
6	Compactness
7	Connectedness
8	Completeness

Exercise Booklet

Handbook

Unit Notes (these appear as part of this Guide)

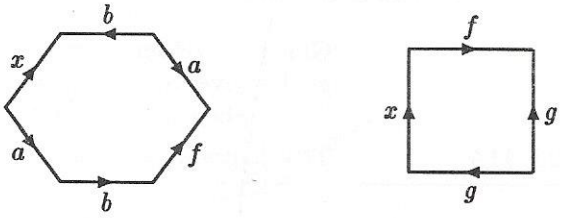
Audio-Cassettes

Tape Booklet

Errata in GT Handbook

Page	*Location*	*Correction*
7	Line −9	'Riemann surface' should be 'Riemann sphere'.
11	Line −6	f should be l.
11	Line −3	l should be h.

Errata in GT Exercise Booklet I

Page	*Location*	*Correction*
12	Question 6(c)	The diagrams should be replaced by the following. 
15	Sect. 4.2, Question 4(d)	This should read '$abcb^{-1}a^{-1}dc^{-1}efe^{-1} = 1$'.
26	Sect. 2.4, Solution 1	The answers for (d) and (e) should be interchanged.
29	Solution 5(a)	This should read '$abca^{-1}c^{-1}b^{-1} = 1$'.
30	Sect. 3.1, Solution 6(b)	Line 1 should read '$xbab = 1$; $x^{-1}cc^{-1}a^{-1} = 1$'. Line 3 should read '$cc^{-1}a^{-1}bab^{-1}$'.
35	Solution 4(b)	There is an '=' missing connecting the first two lines. In fact, all the vertices on the second line, (bc), (cb^{-1}) and (cd), all equal (aa), (ab), etc. This means, on lines 4 and 5, that $N_0 = 2$ and $\chi = -3$. And on the final line of the solution, the canonical form should be $aabbccded^{-1}fgf^{-1} = 1$.
35	Solution 5(b)	Given the correction to Solution 4(b) above, this answer should be: 'A sphere with three crosscaps and two holes.'

Errata in GT Exercise Booklet II

Page	*Location*	*Correction*
7	Question 2(i), line 1	The second a_1 should be a_2, giving the equation '$a_1a_2b_1c_1b_2^{-1} = 1$'.
10	Left-hand column, line 10	'$y_1 = -1$, $y_1 = +1$' should be '$y_1 = -1$, $y_2 = +1$'.
17	Left-hand column, line 10	The first '+' in the numerator should be a '±', so the line should read $= \dfrac{-2\tilde{z}^2 \pm 2\tilde{z}[\tilde{z}^2 + \tilde{z} + 1]^{\frac{1}{2}}}{2(1+\tilde{z})}$. (This may have been corrected already in some versions.)
28	Solution 2(ii)	The smallest region should have a number in it, namely −2.

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Tutorials

As usual, regional arrangements will be made to provide a limited amount of face-to-face and/or telephone tuition for the course. If you are studying the units in a non-standard order, make sure to check with your tutor what material he or she thinks is likely to form the basis of tutorials.

Audio-Cassettes

There are a number of audio-cassette tapes that form an integral part of the course. At various points in the units, you will be instructed to listen to a cassette tape. As often as not, these tapes will themselves contain instructions on what work you should do next, as well as commentaries and advice on the mathematics.

Video-Cassettes

Likewise there are video-cassette tapes which form an integral part of the GT half of the course.

Metric and Topological Spaces Guide

The ideas and methods of general topology permeate modern mathematics. An introduction to this subject is an indispensable part of the training of a pure mathematician. The purpose of this half of the course is to provide such an introduction.

The notion of a topological space, the main object of study in general topology, is introduced by a gradual process of abstraction. In *Unit 2*, we look at real analysis. In *Unit 3*, metric spaces are introduced, and this leads us naturally to topological spaces in *Unit 4*. Three key properties in the applications of general topology are compactness and connectedness of topological spaces, studied in *Units 6* and *7* respectively, and completeness of metric spaces, studied in *Unit 8*.

Unit 1 is a survey of the results from set theory which are needed in later units. We hope that you will find *Unit 1* interesting in its own right; we know that you will appreciate its value as your work for MTS progresses.

The Set Textbook

MTS is based on the set textbook *Introduction to Metric and Topological Spaces*, by W. A. Sutherland, published by Oxford University Press. The revised printing (1981/82) is preferable to the 1975 edition, since some printing errors have been corrected, but the 1975 edition will suffice.

Examples and SAQs

Each unit contains a number of examples, and the sections of each unit end with a number of self-assessment questions (SAQs). Whilst the examples are to be read, the SAQs are for you to do. Some are drill exercises to test understanding of the material in the section, and others are more challenging. We hope that you will be able to find the time to try most of them. Some are marked with an asterisk (*). If you are short of time, you should treat these as examples and read

the solutions. In fact, for all the SAQs, including those for which you are confident you have your own correct solutions and any which you feel you do not have time to attempt, we urge you to read our solutions, since these often contain teaching points. Each unit concludes with a section of revision questions. This section follows the list of Objectives of the unit, and we have tried to test your understanding of each of the key concepts and results in the revision questions.

Workload

Student evaluation of *M335* in 1982 revealed that the workload for MTS was heavier than that for GT. In an attempt to rectify the position, we have decided to deem the last two sections of *Unit 8* non-examinable, with the result that several topics, introduced earlier as prerequisites for this work, can be omitted. The notes which follow will inform you about those sections which you can now safely omit.

Notes on the Units

Below, you will find brief comments on the sections of all the units. We consider it to be crucial that you master completely sections marked with an asterisk. If pressure of time makes it necessary, you might cut corners in some of the others. At the end of the notes on each unit, your attention is drawn to some Examples and SAQs which we feel are particularly important. References to the set book are prefixed by **S**.

MTS Unit 1

1.1 A short section, recalling some familiar ideas, and introducing some notation. The power set of a set is an important new concept.

*1.2 Unions and intersections of infinite collections of sets are defined. De Morgan's Laws (Proposition 3) will be used frequently.

*1.3 This section contains the crucial concept of the inverse image of a set under a map. It is imperative that this new idea be mastered.

1.4 A short section, introducing Cartesian products.

Omit Sections 1.5, 1.6.

Important Examples: 11, 14, 15. Important SAQs: 9, 13, 14, 15.

MTS Unit 2

2.1 The reading passage concentrates on the completeness of $\mathbf{R}$: this property will be used frequently and must be understood. The rational density property is also frequently used.

2.2 Your priorities for this section should be to understand **S** Definition 1.2.2, and to be able to use it. Cauchy sequences (and **S** Theorem 1.2.9) will be of significance in the examinable part of *Unit 8*.

*2.3 This longish section, which is preparatory for Section 2.4, concentrates on **S** Definition 1.3.1. You will need to spend some time studying the definition and the Examples, and in getting practice.

*2.4 Generalization of results about continuous real-valued functions of a real variable is one source of motivation for MTS, so this is a very important section. You must understand **S** Definitions 1.4.3 and 1.4.3′, and be able to apply them. Try *all* the SAQs, if you can find the time, and study the solutions.

Important Examples: 2, 4, 6, 9. Important SAQs: 4, 15, 20.

MTS Unit 3

*3.1 There is a longish reading passage, but the section concentrates on the definition of a metric space (**S** Definition 2.1.2). Give as much time as you can spare to the Examples and SAQs.

3.2 A short section, but it introduces two important ideas (subspaces and products).

*3.3 This section concentrates on the definition of continuity of a function between metric spaces (**S** Definition 2.1.3). You must become familiar with this definition. Parts of Example 5 are used frequently later. Remember the result of SAQ 10.

3.4 This section may be passed over very lightly. You need to know the concept of boundedness of a subset of a metric space, and to study **S** Example 2.2.8.

*3.5 This is the most important section in the unit. It provides the link between metric spaces and topological spaces. Note particularly **S** Propositions 2.3.13, 2.3.17 and 2.3.19, and the results of SAQs 18 and 19.

3.6 Concentrate on topological equivalence. Be sure that you understand and can apply the 'working definition' (TE).

Important Examples: 1, 2, 5, 6, 7, 10, 11. Important SAQs: 1, 2, 16, 17, 21, 24.

MTS Unit 4

*4.1 A very important section: topological spaces are introduced. Spend as much time as possible on the Examples and SAQs, to make this abstract notion come alive.

*4.2 Continuity of functions between topological spaces: again, time spent gaining an understanding of this new idea will be well rewarded. Study the Examples and try all the SAQs.

4.3 This is a short but difficult section on bases. Do not be too worried: the methods, introduced here, of describing or constructing topologies will become familiar with frequent use in later sections.

4.4 A short and easy section on subspaces.

4.5 This section will not be assessed or examined, but the notion of a topological product is important in later work. You should spend enough time on the section to understand this idea.

Important Examples: 1, 3, 4, 5, 6. Important SAQs: 1, 3, 6, 7, 11, 14, 22, 23.

MTS Unit 5

*5.1 This is a straightforward section in which closed sets in a topological space are introduced. Note the result of SAQ 3.

*5.2 The important new ideas of limit point and closure of a set are introduced. Study the Examples carefully.

Omit Sections 5.3, 5.4.

5.5 In this section, we take a quick look at the intuitively appealing Hausdorff condition. Remember that every metric space satisfies the Hausdorff condition, but that there exist non-Hausdorff spaces.

*5.6 Here, homeomorphisms, the 'isomorphisms' of topology, are defined. Try to grasp the significance of the idea of a topological property.

Important Examples: 1, 2, 3, 4, 6, 14, 15. Important SAQs: 2, 8, 29.

MTS Unit 6

6.1 A short section, to motivate the definition of compactness.

*6.2 The important ideas of an open cover of a topological space and of a compact space are introduced. Compactness is one of the major ideas introduced in MTS. Spend as much time as you can spare on the Examples and SAQs in order to appreciate fully the significance of the definitions.

*6.3 The main result of this section is that every bounded closed interval in **R** with the usual topology is compact (Heine–Borel Theorem). You will need to spend some time and effort on understanding its proof.

*6.4 This is a long section, with three reading passages, in which the properties of compact spaces are studied. **S** Propositions 5.4.1, 5.4.2 and 5.5.1, and **S** Theorem 5.9.1, are the most important results.

*6.5 The description of compact sets in Euclidean spaces given here is very important. You must understand it and be able to apply it.
Omit the material from page 45, line −12, to the end of the section, and SAQs 18 and 19.

Omit Section 6.6.

Important Examples: 2, 3, 4, 6, 9. Important SAQs: 1, 2, 3, 9, 11, 15, 16, 23.

MTS Unit 7

*7.1 This section introduces the notion of connectedness. Several equivalent characterizations of connectedness are given. We think that **S** Corollary 6.2.4 gives the most useful one. Remember that the connected subspaces of **R** are the intervals.

*7.2 Properties of connected spaces are found. The most important results are **S** Proposition 6.2.10, **S** Corollary 6.2.13 and **S** Proposition 6.2.18.

7.3 Path-connectedness is defined. For subspaces of Euclidean spaces, this is a geometrically appealing definition. The additional text in this section is important.
Omit **S** Example 6.3.5 and Example 9 of the unit, but note that there *are* connected spaces which are not path-connected.

*7.4 This section treats the components of a topological space. An important technique for homeomorphism problems results from the topological invariance of the number components: see Example 10 and SAQ 16.

7.5 A light-hearted section, in which a famous example is described — can safely be omitted.

Important Examples: 1, 2, 3, 6, 8, 10. Important SAQs: 2, 9, 10, 13, 14, 15, 16, 18, 21.

MTS Unit 8

The objective in *Unit 8* is to study completeness of metric spaces. You will find that many of the ideas introduced in the previous units are employed. Remember that whilst *Unit 8* is not covered by a TMA, those parts of the unit which are identified in this guide are examinable.

*8.1 The idea of a Cauchy sequence is extended to arbitrary metric spaces. This section should be easy reading, but might provide some surprises!

8.2 You may pass lightly over this section, noting only the definition of sequential compactness and the fact that a metric space is compact if and only if it is sequentially compact.

*8.3 Metric spaces of real-valued functions on a bounded closed interval, introduced in *Unit 3*, are studied more extensively. If possible, you should study all of the additional text and the examples in detail. If you have met the idea of uniform convergence before, we feel that you will see that this is the 'right' way to treat it.

*8.4 Here, we meet the concept of completeness of a metric space. This is one of the major ideas of modern mathematics. The main results of the section are S Propositions 9.2.3 and 9.2.4, S Example 9.2.6, and Example 10 of the unit. Do not worry too much about the details of Example 12.

Omit Sections 8.5 and 8.6.

Important Examples: 1, 2, 3, 7, 10, 11. Important SAQs: 1, 4, 12, 13, 26.

Supplement to Unit 2 Metric and Topological Spaces

FOR STUDENTS WHO STUDIED *M203*
(Introduction to Pure Mathematics)
IN OR AFTER 1987

You may know that an *M203* re-write was presented for the first time in 1987. Before 1987, the *M203* approach to the fundamental concepts of analysis, whilst equivalent to that reviewed by Sutherland, was apparently rather different. To bridge the two approaches, a number of notes, indicated by the symbol *M203* in the left-hand margin, are included in MTS *Unit 2*. These notes were intended for pre-1987 *M203* students, and if you studied *M203* in or after 1987, you should *IGNORE* the notes with the marginal mark *M203*. Really, little more needs to be said as the present *M203* treatment of analysis is very close to that reviewed by Sutherland. Some small apparent differences are reconciled in *Unit 5* of *M203* Analysis Block B; if you did not have time earlier to study this non-assessed unit, you are strongly recommended to study Sections 2.2 and 2.4 in conjunction with MTS *Unit 2*. For reassurance only, there follows a replacement for each note in MTS *Unit 2* with the marginal mark *M203*.

Page 38, Note (i)

The real number system is the subject of *Unit 1* of *M203* Analysis Block A. The properties of addition, subtraction, multiplication and division of real numbers are listed in Section 1.5. The way in which equalities work is discussed in Section 2.1, and the interaction of inequalities with addition and multiplication is described in Section 3.

Page 39, Note (v)

S Axiom 1.1.4 is precisely the Least Upper Bound Property of **R** introduced in Section 4.3 of *Unit 1* of *M203* Analysis Block A.

Page 40, Note (viii)

The Archimedean Property (**S** Proposition 1.1.6) and the Density Property (**S** Corollary 1.1.7 and following Remark) are discussed in Section 1.4 of *Unit 1* of *M203* Analysis Block A.

Page 44, Note (ii)

S Definition 1.2.2 is the Equivalent Definition of convergence of a sequence, considered in Section 3.1 of *Unit 2* of *M203* Analysis Block A.

Page 45, Note (vii)

The Cauchy convergence criterion is not considered in *M203*. **S** Theorem 1.2.9 should be regarded as a supplement to the knowledge of convergence of sequences obtained in *M203*. Cauchy sequences will be considered in a more general setting in MTS *Unit 8*, where it will be seen that **S** Theorem 1.2.9 expresses a very important property of **R**.

Page 51, Note (i)

The definition of the limit of a function at a point given in Section 3.1 of *Unit 5* of *M203* Analysis Block A is the condition, equivalent to **S** Definition 1.3.1, stated as (ii) in **S** Lemma 1.3.5. The proof of **S** Lemma 1.3.5 is given in Section 2.4 of *Unit 5* of *M203* Analysis Block B.

Page 52, Note (vii)

The Combination Rules for limits of functions (the analogue for limits of functions of **S** Proposition 1.2.11) are stated in Section 3.3 of *Unit 5* of *M203* Analysis Block A.

Page 57, Note (i)

S Definition 1.4.3′ is the ε-δ definition of continuity discussed in Section 2.2 of *Unit 5* of *M203* Analysis Block B. Its equivalence with **S** Definition 1.4.3 is proved in Theorem 2 of that section.

Page 58, Note (v)

The I.V.P. for a continuous function $f : [a, b] \longrightarrow \mathbf{R}$ is discussed in Section 4.1 of *Unit 4* of *M203* Analysis Block A and it is the subject of an associated *M203* television programme. As pointed out in **S**, the I.V.P. for a continuous function $f : \mathbf{R} \longrightarrow \mathbf{R}$ is a consequence of the property of **R** called connectedness. This notation will be introduced in MTS *Unit 7* where it will be shown also that each interval $[a, b]$ is connected.

Geometric Topology Guide

We are all familiar with surfaces in everyday life: the surface of a piece of paper, the surface of a ball, the surface of a cube, and so on. Then there are other, more 'mathematical' surfaces, like the projective plane, important in geometry. Can we bring all these surfaces under one mathematical umbrella?

Well, we can, and as a result we can answer such questions as: 'What does a conic such as $y^2 = x$ look like when we allow y and x to be complex numbers?' We can also give a complete description of all possible surfaces, and say which of them can actually be modelled in Euclidean three-dimensional space. As well as being interesting for their own sake, these results have important applications in differential equations (which we try to illustrate in the final GT unit, developing the idea of flows which you may have met in *M203*) and in many other parts of modern mathematics. In fact, the study of manifolds, the higher-dimensional analogue of surfaces, has become one of the most important and active parts of mathematics during this century.

Unlike MTS, GT is firmly rooted in geometric, visual ideas. There are many pictures in the units and on the video-cassettes to start off the investigations (so that watching the video-cassettes is very important). We then introduce a number of mathematical concepts to help us talk about surfaces in an accurate, precise way. Most of GT is concerned first with these basic concepts, and secondly with using them to solve problems. There are a few important theorems, such as the central result of this half of the course: the classification theorem for surfaces. But we are concerned not so much with the proofs of these theorems as with their applications. Hence the Examples and the SAQs are most important, and you should pay very careful attention to them. There are also plenty of drill exercises in the *Exercise Booklets*; we do not expect you to do all of them, but you should be able to tackle a representative selection. They should provide you with very useful practice for the TMAs.

Another part of GT, which contains many worked examples, is Section 8.3 of *Unit 8*. This is designed to be a 'revision module' which brings together all the main ideas of this half of the course in a series of examples. We hope it will help you revise for the exam if you work systematically through it as part of your work in *Unit 8*.

The Set Textbook

We have provided you with a copy of the set textbook *Elementary Topology*, by Donald W. Blackett (Academic Press, 2nd edition, 1982). About 90 pages of this book, and some of the problems, are set for study. (When the course first came out, students had to buy their own copy of the book. The book is now out of print so the Open University obtained permission to copy it for you.) Please note that the photocopy excludes Chapters 6 and 7 of the original book, as these are not referred to within the course, and the Bibliography and Index.

Video- and Audio-cassettes

The GT half of the course was originally accompanied by eight TV programmes. These programmes have now been put onto video-cassettes which are sent to you as part of the course. We hope that your study will benefit from the opportunity to watch them when it is convenient for you, and to replay them — there are many helpful and beautiful graphics worth a second look! Each unit has one section centred on the TV programme. The work included in the programme is not printed in full in the text: television is an ideal medium for explaining many of the ideas in geometric topology, and some of these ideas are much more difficult

to explain in print alone. The audio-cassette Tapes A1 and A2 explain what work is necessary before you watch the video, and also give a 'TV overview', or summary of the programme, which you should work through as soon as possible after the programme. You can also use these overviews for revision. Have a look, preferably NOW, at Section 1.4 of GT *Unit 1*, which describes the function of the TV and audio sections.

Relation to MTS

The Geometric Topology half of the course is self-contained, although both halves of *M435* both deal with topological ideas. GT deals with a part of topology different from the subject of MTS, and concentrates on objects in Euclidean space which arise from geometry and simple geometrical constructions.

Notes on the Units

In what follows, we describe briefly the contents of each section of the units, and indicate their relative importance by starring the most important sections. If you are short of time, you can cut corners in the unstarred sections: concentrate on the worked examples and SAQs.

GT Unit 1

∗1.1 Introduction to surfaces and the boundary of a surface.

∗1.2 Coordinate systems on the sphere and torus; stereographic projection. A technical section, needed later, but involving no hard ideas, only algebraic formulae.

∗1.3 Homeomorphisms and other topological ideas. Concentrate on the definitions and examples here, and do not spend too long on a first reading.

∗1.4 (TV section) Some basic definitions: formal definition of a surface, boundary of a surface, subdivision of a surface; the Euler characteristic. This section states the classification theorem, which will be our goal for the whole of the first half of GT. The follow-up exercises are very important, but, we hope, straightforward.

GT Unit 2

2.1 Further work on the Euler characteristic, applying it to map colourings and regular subdivisions. The applications themselves are not so important, but the practice in techniques is valuable.

∗2.2 More about the boundary of a surface: the boundary has to be a collection of circles. Important application: how to prove things about Möbius bands. An interesting digression on paper models.

∗2.3 Some surfaces cannot be made in $\mathbb{R}^3$. The projective plane $\mathbb{RP}^2$ and the Klein bottle are two examples: how can we study them?

∗2.4 (TV section) Orientability. This is the third concept needed to enable us to classify surfaces (the other two are Euler characteristic and boundary). How to verify orientability in practice, and why every non-orientable surface contains a Möbius band.

2.5 One-sided and two-sided embeddings. A very short peripheral section.

GT Unit 3

*3.1 Every surface can be divided up into a number of polygons (topological discs) which have to be re-glued along the edges to make the surface. This leads to a method of describing a surface by simple algebraic equations called edge equations.

*3.2 How simple 'cut and paste' operations on surfaces correspond to simple algebraic transformations on the edge equations. This will be used later in the classification theorem.

3.3 (TV section) Various descriptions of the projective plane — including the ones you may have met in projective geometry in *M203* and in *M101* — are examined, and we see how, topologically speaking, all these descriptions are equivalent. Very useful background, some of which is needed for *Units 5* and *6* — mainly the description of the 'antipodal model' of $\mathbb{RP}^2$.

3.4 (Non-assessed; audio-cassette B) This section extends the definition of a continuous function from Euclidean spaces to general surfaces, and relates this to both the *M203* definition of continuity and other definitions. Necessary if you want to fill in all the details of proofs, but not really about surfaces as such.

GT Unit 4

*4.1 More about the algebra of edge equations. An essential computational section.

*4.2 Combinatorial invariants of a surface, and how to calculate them.

*4.3 (TV section) The connected sum construction, and how to make any surface out of spheres, discs, tori and projective planes. This section puts all the preceding work into a uniform context, and makes it easier to remember, but it is not used again in later units (though it is assessable).

4.4 (Peripheral) Some remarks about the triangulation theorem and other hard theorems, with suggestions for further reading.

GT Unit 5

*5.1 Complex loci, that is, the study of complex solutions of equations like $w^2 = z$. The space $\mathbb{C} \cup \{\infty\}$ is a topological sphere, and we can use the methods of *Units 1–4* to study complex conics (and other complex loci), if we make them compact by adding the point at infinity to ordinary complex numbers.

*5.2 (TV section) Loci of complex quadratics. This section studies the locus $w^2 = z$ in detail, for complex values of w and z.

*5.3 This extends the work of Section 5.2 to any complex conic, the general quadratic locus. A hard section, but one which gives a very satisfactory application of all the preceding work, and explains the link between geometry and topology.

5.4 Worked examples.

5.5 (Non-assessed; audio-cassette B) This extends the definition of continuity to the space $\mathbb{C} \cup \{\infty\}$. The remarks of Section 3.4 apply to this section also.

5.6 (Non-assessed appendix) A proof is sketched that a complex locus defines a surface away from the singular points.

GT Unit 6

*6.1 One description of the projective plane (the antipodal model) shows that there is a continuous function $S^2 \longrightarrow \mathbb{RP}^2$ such that exactly two points of the sphere map to every point of $\mathbb{RP}^2$. This leads to the idea of a covering surface of a surface, and several other examples are discussed.

*6.2 (TV section) Various methods to construct covering surfaces are described, and it is shown how several examples we have already met are covering surfaces. The question of which surfaces can cover a given surface is examined in detail.

6.3 The algebraic method of constructing covering surfaces by using edge equations is discussed further, as are the Riemann surfaces of some simple polynomials. A harder section, included for its importance in other branches of mathematics, particularly complex analysis and advanced geometry.

6.4 More complex loci, described in examples.

Summary of Units 5 and 6.

GT Unit 7

Apart from the definition of $\mathbb{C} \cup \{\infty\}$ (used in Sections 7.3 and 7.4) and the locus $w^2 = z$ (TV5), this unit is independent of *Units 5* and *6*.

*7.1 (TV section) Winding number of a loop about a point, and an introduction to the Brouwer degree.

*7.2 Methods of calculating winding numbers.

7.3 Proof of the Brouwer degree theorem.

7.4 Application of the Brouwer degree theorem.

GT Unit 8

This unit is in two parts: Sections 8.1 and 8.2 form the first part, and this part is independent of *Units 5* and *6*, and uses only the definition of winding number from *Unit 7*.

8.1 Revision of plane flows, and a description of linear flows near a singular point. (Revision of material in *M203*.)

8.2 Flows on surfaces. Using ideas similar to the idea of winding number, the index of a singular point of a flow is defined. A remarkable theorem, due to Poincaré, relates the Euler characteristic of a surface to the index sum of singular points of a flow on the surface. Thus ***analytical*** problems, such as differential equations, can be subject to genuine topological restrictions.

A simple question based on this work will be included on the exam paper, but Sections 8.1 and 8.2 will not be assessed in the TMAs.

*8.3 Revision section.

Errata and Addenda: Metric and Topological Spaces

Errata in Sutherland

All the following errors have been corrected in the most recent printing of Sutherland.

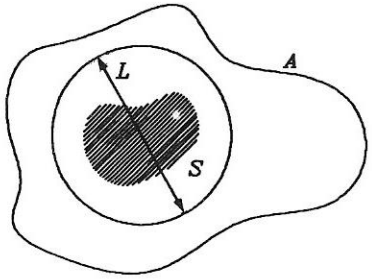
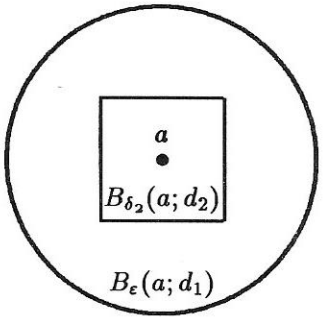
Page	*Location*	*Correction*
4	Proof of Corollary 1.1.7	This should read as follows. 'Suppose $x < y$, and first suppose $x \geq 0$. Since $y - x > 0$, by 1.1.6 there is an n in $\mathbf{N}$ satisfying $n > 1/(y-x)$ so $1/n < y - x$. Let $M = \{m \in \mathbf{N} : m/n > x\}$. By 1.1.6, $M \neq \varnothing$, otherwise nx would be an upper bound for $\mathbf{N}$. Hence, since $M \subset \mathbf{N}$, M contains a least integer v. This means $v/n > x$, $(v-1)/n \leq x$. Hence $x < v/n \leq x + 1/n < x + (y - x) = y$ and v/n is a suitable rational number. Now suppose $x < 0$. If $y > 0$ then 0 is a rational such that $x < 0 < y$. If $y \leq 0$, the previous case supplies a rational r such that $-y < r < -x$, so $x < -r < y$.'
6	Example 1.2.1(f)	The formula for s_n should read '$s_n = \frac{1}{2}(s_{n-1} + s_{n-2})$'.
12	Examples 1.3.4	The domain of both f and g should be $\mathbf{R} - \{0\}$.
15	Line 3	'$\lim_{x \Sigma a+}$' and '$\lim_{x \Sigma a-}$' should read '$\lim_{x \to a+}$' and '$\lim_{x \to a-}$'.
16	Exercise 11	On line 2 insert 'positive' before 'integer n'. On last line add '$= a$.' at end.
20	Lines -1 and -2	Delete these lines.
24	Line 19	For 'in $\mathbf{N}$' read 'n in $\mathbf{N}$'.
42	Line 6	The title of Section 2.5 should be 'Continuity' not 'Continuo'.
60	Example 3.7.5	Replace Example 3.7.5 and the preceding paragraph (beginning with 'It is worthwhile ...') with the following. 'One of these involves the useful concept in the next definition. DEFINITION 3.7.5. A *neighbourhood* of a point x in a topological space T is a subset V of T such that $x \in U \subset V$ for some open subset U of T. Some textbooks require neighbourhoods to be open, but our definition is more commonly used.'
62	Proposition 3.7.15(b)	This should read '$H \subset K \Rightarrow Cl(H) \subset Cl(K)$'.
70	Exercise 22	Replace this exercise with the following. 'Prove that a finite subset of a metric space has no limit points.'
76	Line -4	Replace 'open balls' with 'balls open in A'.
92	Line 7	'Exercise 8' should be 'Exercise 7'.
105	Exercise 3(iv)	For '$([0,1] \times \{1\})$' read '$(\mathbf{R} \times \{1\})$'.
109	Proof of Lemma 7.2.5, lines 2–7	Replace the passage 'First, we may as well assume ... for every n, $x_n \neq x$.' with the following. 'First, S is infinite by Exercise 3.9.22. We may assume that x_n never equals x. For otherwise, delete all the occurrences of x from (x_n): the result is still a sequence since S is infinite, and its set of members $S - \{x\}$ still has x as a limit point.'

Page	*Location*	*Correction*
122	Exercise 3(iii)	f should be defined by '$f_n(x) = n^{\frac{1}{2}}x(1-x)^n$'.
164	Solution to Exercise 13	Replace the solution with the following. 'If $U \in \mathscr{T}_H$, then $U = V \cap H$, for some V in $\mathscr{T}$. For $x \in U$, $B_\varepsilon(x;d) \subset V$ for some $\varepsilon > 0$, so $B_\varepsilon(x;d_H) \subset U$ and U is d_H-open. If U is d_H-open, then for each x in U, $B_{\varepsilon(x)}(x;d_H) \subset U$ for some $\varepsilon(x) > 0$, and $U = H \cap V$ where $V = \bigcup_{x \in U} B_{\varepsilon(x)}(x;d)$ is in $\mathscr{T}$.'
165	Solution to Exercise 24	This should read '$H = (0,2) \cup (3,4)$, $K = (1,3)$'.

Errata in MTS Units 1–4

Page	*Location*	*Correction*
13	Solution to SAQ 6(a)	The final sentence should read 'Therefore $A \subset B$.'.
19	Solution to SAQ 11, line 1	'$f^{-1}([4,9)]$' should be '$f^{-1}([4,9])$'.
28	Line 9	'$(q_1(f_1 \times f_2))(x)$' should be '$q_1((f_1 \times f_2)(x))$'.
47	Line −4	For 'width' read 'half-width'.
50	Solution to SAQ 9(b)	Replace with the following. 'Since $s_{n+1} - s_n = \dfrac{1}{(n+1)^3} > 0$, the sequence is monotonic increasing. Next, since $n \leq 2^n - 1$ for all positive integers n, we have $s_n \leq s_{2^n-1}$ $\leq 1 + \left(\dfrac{1}{2^3} + \dfrac{1}{3^3}\right) + \left(\dfrac{1}{4^3} + \cdots + \dfrac{1}{7^3}\right) + \cdots + \left(\dfrac{1}{(2^{n-1})^3} + \cdots + \dfrac{1}{(2^n-1)^3}\right)$ $\leq 1 + \dfrac{2}{2^3} + \dfrac{4}{4^3} + \cdots + \dfrac{2^{n-1}}{(2^{n-1})^3}$ $= 1 + \dfrac{1}{4} + \dfrac{1}{4^2} + \cdots + \dfrac{1}{4^{n-1}}$ $< \sum_{n=1}^{\infty} \dfrac{1}{4^{n-1}} = \dfrac{4}{3}$. Thus the sequence (s_n) is a monotonic increasing sequence which is bounded above and so converges by **S** Theorem 1.2.6.'
60	SAQ 20(v), line 3	For '$f : x \longmapsto \dfrac{1}{f(x)}$' read '$\dfrac{1}{f} : x \longmapsto \dfrac{1}{f(x)}$'.
62	Last line	The denominator of the right-hand side of the equation should be '$\lvert f(x)\rvert\,\lvert f(a)\rvert$'.
64	SAQ 24	The rule of f should read $f : x \longmapsto \begin{cases} 0, & \text{if } x = 0, \\ \dfrac{1}{x}, & \text{if } x \neq 0. \end{cases}$

Page	*Location*	*Correction*
77	Lines 7–9	Replace with the following. 'Let $\delta_1 = \min(2, \delta)$. Let $b = (b_1, -\frac{1}{2}\delta_1)$ where b_1 is the positive square root of $1 - \frac{1}{4}\delta_1^2$, so $b_1 = +\sqrt{1 - \frac{1}{4}\delta_1^2}$. As $0 < \delta_1 \leq 2$, the equations $\cos t = \sqrt{1 - \frac{1}{4}\delta_1^2}, \quad \sin t = -\frac{1}{2}\delta_1,$ have a joint solution $t \in [0, 2\pi)$, so that $b \in S^1$. Checking that $d_\infty(b, a) < \delta$, where $d_\infty\left(\left(\sqrt{1 - \frac{1}{4}\delta_1^2}, -\frac{1}{2}\delta_1\right)(1,0)\right) = \text{Max}\left(\left|\sqrt{1 - \frac{1}{4}\delta_1^2} - 1\right|, \left|-\frac{1}{2}\delta_1 - 0\right|\right),$ we have $\left(1 - \sqrt{1 - \frac{1}{4}\delta_1^2}\right)\left(1 + \sqrt{1 - \frac{1}{4}\delta_1^2}\right) = 1^2 - \left(\sqrt{1 - \frac{1}{4}\delta_1^2}\right)^2$ $= 1 - (1 - \frac{1}{4}\delta_1^2)$ $= \frac{1}{4}\delta_1^2,$ so $1 - \sqrt{1 - \frac{1}{4}\delta_1^2} = \dfrac{\frac{1}{4}\delta_1^2}{1 + \sqrt{1 - \frac{1}{4}\delta_1^2}}$ $< \frac{1}{4}\delta_1^2 \quad \left(\text{as } \sqrt{1 - \frac{1}{4}\delta_1^2} > 0\right)$ $= \delta_1 \times \dfrac{\delta_1}{4}$ $< \delta_1 \quad \left(\text{as } \delta_1 < 2, \text{ so } \dfrac{\delta_1}{4} < 1\right)$ $\leq \delta \quad (\text{as } \delta_1 \leq \delta).$ Hence $\left|\sqrt{1 - \frac{1}{4}\delta_1^2} - 1\right| < \delta$, while $\left| -\frac{1}{2}\delta_1 - 0\right| = \frac{1}{2}\delta_1 < \delta_1 \leq \delta$. Thus $d_\infty(a, b) < \delta$. As $b \in S^1$ and $d_\infty(a, b) < \delta$, our original discussion tells us that $0 \leq h(b) < \pi$. Now $h = g^{-1}$, so $g(h(b)) = b = \left(\sqrt{1 - \frac{1}{4}\delta_1^2}, -\frac{1}{2}\delta_1\right)$. However, $g(h(b))$ also equals $(\cos h(b), \sin h(b))$, so $\sin h(b) = -\frac{1}{2}\delta_1$. As $0 \leq h(b) < \pi$, we must have $\sin h(b) \geq 0$, while $-\frac{1}{2}\delta_1 < 0$. Thus the equation $\sin h(b) = -\frac{1}{2}\delta_1$ cannot be satisfied, so we have our contradiction.'
77	Line −2	Insert 'modulus of the' before 'difference'.

Page	*Location*	*Correction*
78	Note (iv) diagram	L looks as though it measures a diameter of the set S as drawn. The following diagram may be more helpful. 
85	Diagram	The names of the sets in the right-hand diagrams should be swapped. Thus the diagram should look as follows. 

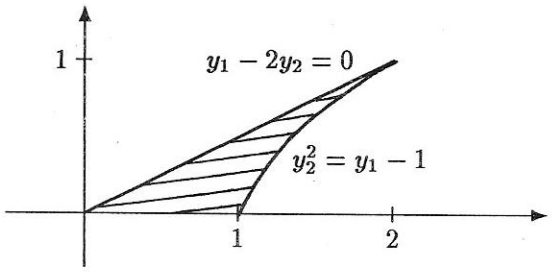
Errata in MTS Units 5–8

Page	*Location*	*Correction*
39	Reading instructions	These should be 'READ **S** Section 5.4, page 82, up to (but *not* including) Definition 5.4.3.'
41	Example 7, line 12	The set E should be $\left[1-\frac{1}{m},1\right)$ rather than $\left[\frac{1}{m},1\right)$.
49	Reading instructions	These should be 'READ all of **S** Section 5.8.'

Erratum in MTS Tape Booklet

Page 8: The frames on page 8 accompany Side 2 of Tape 2.

Erratum in MTS Exercise Booklet

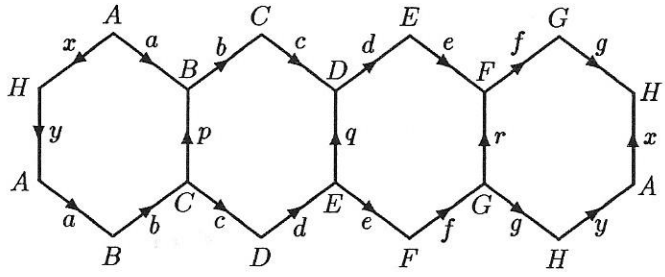
Page	*Location*	*Correction*
11	Lines 15, 16	In the solution to Question 1.3(ii)(a) it is asserted that if $g : \mathbf{R}^2 \to \mathbf{R}^2$ is given by $g(x_1, x_2) = (x_1^2 + x_2^2, x_1 x_2)$, $(x_1, x_2) \in \mathbf{R}^2$, then $g([0,1] \times [0,1]) = [0,2] \times [0,1]$. This is incorrect. If $0 \leq x_1, x_2 \leq 1$, then $0 \leq x_1^2 + x_2^2 \leq 2$ and $0 \leq x_1 x_2 \leq 1$, so that $g([0,1] \times [0,1]) \subset [0,2] \times [0,1]$ but we cannot assert that every point of $[0,2] \times [0,1]$ is the image under g of a point of $[0,1] \times [0,1]$. For example, if $x_1^2 + x_2^2 = 0$ then $x_1 x_2 = 0$, so that $(0,1)$ is not the image under g of any point of $[0,1] \times [0,1]$. In fact, $g([0,1] \times [0,1]) = C$ where $C = \{(y_1, y_2) \in \mathbf{R}^2 : 0 \leq y_2 \leq 1, 2y_2 \leq y_1 \leq y_2^2 + 1\}$. The subset C takes the following form.  If $0 \leq x_1, x_2 \leq 1$, then $0 \leq x_1 x_2 \leq 1$, and since $(x_1 - x_2)^2 \geq 0$, it follows that $2x_1 x_2 \leq x_1^2 + x_2^2$. Also, if $0 \leq x_1, x_2 \leq 1$ and $x_1 x_2 = 0$, then $0 \leq x_1^2 + x_2^2 \leq 1$, whilst if $x_1 x_2 = b > 0$ then $x_1 \geq b$ (as $x_2 \leq 1$), so that $$(b^2+1) - (x_1^2 + x_2^2) = (b^2+1) - \left(x_1^2 + \frac{b^2}{x_1^2}\right) = \left(1 - \frac{b^2}{x_1^2}\right)(1 - x_1^2) \geq 0.$$ Thus $g([0,1] \times [0,1]) \subset C$. Now let $(a,b) \in C$ so that $0 \leq b \leq 1$ and $2b \leq a \leq b^2 + 1$. If we put $c = (a^2 - 4b^2)^{1/2}$ and $$x_1 = \left(\frac{a+c}{2}\right)^{1/2}, \quad x_2 = \left(\frac{a-c}{2}\right)^{1/2},$$ then $0 \leq x_1, x_2 \leq 1$ and $x_1^2 + x_2^2 = a, \quad x_1 x_2 = b,$ so that $g(x_1, x_2) = (a,b)$. Therefore $g([0,1] \times [0,1]) = C$.

Errata and Addenda: Geometric Topology

Errata in Blackett

Page	*Location*	*Correction*
64	Operation 3, line 1	Insert 'distinct' before 'letters u and v'.
125	Line −12	This should read $r = \left\lvert \frac{p_1(z_1)}{q_1(z_1)} \right\rvert$.

Errata in GT Units 1,2

Page	*Location*	*Correction*
52	Solution to SAQ 14(i)	The solution is incorrect, as e.g. only 2 edges meet at G while 3 edges meet at E. A correct solution is as follows.  Glueing together the edges labelled a, then those labelled b, and so on up to g, this gives a cylinder with edges oriented in such a way as to give a Klein bottle when they are matched up.
55,56	Frames 11, 11A	The diagrams for Frames 11 and 11A should be interchanged.

Errata in GT Units 3,4

Page	*Location*	*Correction*
6	Diagram	Delete the dotted red lines on the torus.
16	Bottom diagram	The left-hand torus should have a hole — i.e. the interior of c should be deleted.

Errata in GT Units 5,6

Page	*Location*	*Correction*
14	Frame 2, lines 4–6	θ and ϕ should be interchanged, so that the lines read as follows. 'then $w^2 = s^2e^{2i\phi} = re^{i\theta}$, which implies $s^2 = r$ and $2\phi = \theta + 2n\pi$, so $s = \sqrt{r}$ and $\phi = \dfrac{\theta}{2} + n\pi$.'
24	SAQ 4, line 1	'For what value' should be 'For what finite value'.
30	Line 17	For 'Riemann sphere' read 'Riemann surface'.
41	Line 5	The example should refer to the equation $y^2 = x^2(x+1)$, to match the diagram.
41	Line 9	This should start with '$g(x,y) = y^2 - x^2(x+1) = 0 \ldots$'.
41	Line 11	This should read '$\dfrac{\partial g}{\partial x} = -3x^2 - 2x \neq 0$ unless $x = 0$ or $-\frac{2}{3}$'.
41	Lines 14 and 15	On each line replace '$x = \frac{2}{3}$' with '$x = -\frac{2}{3}$'.
49	Definition (i), line 2	For 'f maps' read 'p maps'.
49	Definition (ii), line 2	For 'P restricted' read 'p restricted'.
52	Frame 4, cloud, line 2	'$\tilde{S} \xrightarrow{n} \tilde{S}$' should be '$\tilde{S} \xrightarrow{n} S$'.
58	Frame 12, line 2	S should be $\tilde{S}$.
68	Lines −2,−1	This should read '$\chi_S = -4 = 2\chi_{S_y} - 4$ gives $\chi_{S_y} = 0$ (i.e. S_y is a torus or a Klein bottle)'.
69	Line 1	S should be S_y.
71	Line −1	p_1 should be P_1.